Master of Information and Data Science
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Professor Hany Farid
University of California, Berkeley

Fourier Series and Transform

Consider a 1-D discretely sampled signal of length m:

$$f(x) \;=\; \begin{pmatrix} 1 & 2 & 4 & 5 & 3 & 0 & \ldots & 7 \end{pmatrix}. \tag{1}$$

Although not made explicit, such a signal is represented with respect to a basis consisting of the canonical vectors (or unit impulses) in $\mathcal{R}^m$. That is, the signal is represented as a weighted sum of the basis vectors:
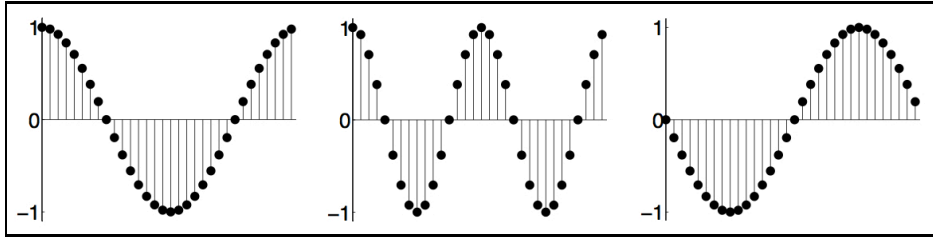
$$
\begin{aligned}
f(x) \;=\;\; & 1 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \\
+\;\; & 2 \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \\
+\;\; & 4 \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \\
+\;\; & \ldots \\
+\;\; & 7 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}.
\end{aligned} \tag{2}
$$

This can be written more compactly as:

$$f(x) \;=\; \sum_{k=0}^{m-1} a_k b_k(x), \tag{3}$$

where $b_k(x)$ are the canonical basis vectors, and:

$$a_k \;=\; \sum_{l=0}^{m-1} f(l) b_k(l). \tag{4}$$

Figure 1: 1-D Fourier basis.

In the language of linear algebra, the weights a_k are simply an inner product between the signal $f(x)$ and the corresponding basis vector $b_k(x)$.

A signal (or image) can be represented with respect to any of a number of different basis vectors. A particularly convenient and powerful choice is the Fourier basis. The Fourier basis consists of sinusoids of varying frequency and phase, Figure 1. Specifically, we seek to express a periodic signal as a weighted sum of the sinusoids:

$$f(x) \;=\; \frac{1}{m} \sum_{k=0}^{m-1} c_k \cos\left(\frac{2\pi k}{m} x + \phi_k\right), \tag{5}$$

where the frequency of the sinusoid is $\omega_k = 2\pi k/m$, the phase is ϕ_k, and the weighting (or amplitude) of the sinusoid is c_k. The sinusoids form a basis for the set of periodic signals. That is, any periodic signal can be written as a linear combination of the sinusoids. This expression is referred to as the Fourier series.

Note that this basis is not fixed because the phase term, ϕ_k, is not fixed, but depends on the underlying signal $f(x)$. This can become problematic when comparing the Fourier representation of two or more signals which will be expressed with respect to different basis vectors and therefore will no longer be comparable. It is, however, possible to rewrite the Fourier series with respect to a fixed basis of zero-phase sinusoids. With the trigonometric identity:

$$\cos(A + B) \;=\; \cos(A)\cos(B) - \sin(A)\sin(B), \tag{6}$$

the Fourier series of Equation (5) may be rewritten as:

$$
\begin{aligned}
f(x) &= \frac{1}{m} \sum_{k=0}^{m-1} c_k \cos(\omega_k x + \phi_k) \\
&= \frac{1}{m} \sum_{k=0}^{m-1} c_k \cos(\phi_k) \cos(\omega_k x) + c_k \sin(\phi_k) \sin(\omega_k x) \\
&= \frac{1}{m} \sum_{k=0}^{m-1} a_k \cos(\omega_k x) + b_k \sin(\omega_k x).
\end{aligned}
\tag{7}
$$

The basis of cosine and sine of varying frequency is now fixed. Notice the similarity to the basis representation in Equation (3): the signal is being represented as a weighted sum of a basis. We've simply replaced the canonical basis with a sine and cosine basis.

The Fourier series tells us that a signal can be represented in terms of the sinusoids. The Fourier transform tells us how to determine the relative weights a_k and b_k:

$$
a_k = \sum_{l=0}^{m-1} f(l) \cos(\omega_k l) \quad \text{and} \quad b_k = \sum_{l=0}^{m-1} f(l) \sin(\omega_k l).
\tag{8}
$$

As in Equation (4), these Fourier coefficients are determined from an inner product between the signal and corresponding basis.

A more compact notation is often used to represent the Fourier series and Fourier transform which exploits the complex exponential and its relationship to the sinusoids:

$$
e^{i\omega x} = \cos(\omega x) + i \sin(\omega x),
\tag{9}
$$

where i is the complex value $\sqrt{-1}$. With this complex exponential notation, the Fourier series and transform take the form:

$$
f(x) = \frac{1}{m} \sum_{k=0}^{m-1} c_k e^{i\omega_k x} \quad \text{and} \quad c_k = \sum_{l=0}^{m-1} f(l) e^{-i\omega_k l},
\tag{10}
$$

where $c_k = a_k - i b_k$. This notation simply bundles the sine and cosine terms into a single expression.
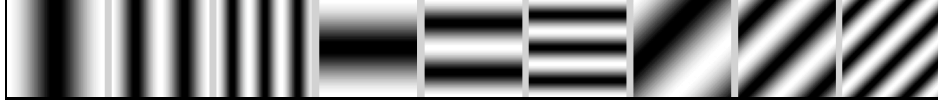
Figure 2: 2-D Fourier basis.

The Fourier coefficients c_k are complex valued. These complex valued coefficients can be analyzed in terms of their real and imaginary components, corresponding to the cosine and sine terms. This can be helpful when exploring the symmetry of the underlying signal $f(x)$, as the cosine terms are symmetric about the origin and the sine terms are asymmetric about the origin. These complex valued coefficients can also be analyzed in terms of their magnitude and phase. Considering the complex value as a vector in the real-complex space, the magnitude and phase are defined as:

$$|c_k| = \sqrt{a_k^2 + b_k^2} \qquad \text{and} \qquad \angle c_k = \tan^{-1}(b_k/a_k). \qquad (11)$$

The magnitude describes the overall contribution of a frequency in constructing a signal, and the phase describes the relative position of each frequency.

In 2-D: In two dimensions, an $m \times m$ image can be expressed with respect to two-dimensional sinusoids:

$$f(x,y) \;\; = \;\; \frac{1}{m^2} \sum_{k=0}^{m-1} \sum_{l=0}^{m-1} c_{kl} \cos(\omega_k x + \omega_l y + \phi_{kl}), \qquad (12)$$

with:

$$c_{kl} \;\; = \;\; \sum_{u=0}^{m-1} \sum_{v=0}^{m-1} f(u,v) \cos(\omega_k u + \omega_l v + \phi_{kl}), \qquad (13)$$

The 2-D Fourier basis consist of sinusoidal gratings of varying orientation and frequency, Figure 2. From left to right are three vertically oriented basis of increasing frequency ($\omega_k > 0$ and $\omega_l = 0$), three horizontally oriented basis of increasing frequency ($\omega_k = 0$ and $\omega_l > 0$), and three oblique basis of increasing frequency ($\omega_k > 0$ and $\omega_l > 0$).

As with the 1-D Fourier basis, the 2-D Fourier basis can be expressed with respect to a fixed basis as:

$$f(x,y) = \frac{1}{m^2} \sum_{k=0}^{m-1} \sum_{l=0}^{m-1} a_{kl} \cos(\omega_k x + \omega_l y) + b_{kl} \sin(\omega_k x + \omega_l y), \qquad (14)$$

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where,

$$a_{kl} = \sum_{u=0}^{m-1}\sum_{v=0}^{m-1} f(u,v)\cos(\omega_k u + \omega_l v) \tag{15}$$

$$b_{kl} = \sum_{v=0}^{m-1}\sum_{v=0}^{m-1} f(u,v)\sin(\omega_k u + \omega_l v). \tag{16}$$

As with the 1-D Fourier basis and transform, the sine and cosine terms can be bundled using the complex exponential:

$$f(x,y) = \frac{1}{N^2}\sum_{k=0}^{m-1}\sum_{l=0}^{m-1} c_{kl}e^{i(\omega_k x + \omega_l y)} \tag{17}$$

$$c_{kl} = \sum_{u=0}^{m-1}\sum_{v=0}^{m-1} f(u,v)e^{-i(\omega_k u + \omega_l v)}, \tag{18}$$

where $c_{kl} = a_{kl} - b_{kl}$. The Fourier transform c_{kl} is often denoted as $F(\omega_k, \omega_l)$.

Because the Fourier basis are periodic, the Fourier representation is particularly useful in discovering periodic patterns in a signal that might not otherwise be obvious when the signal is represented with respect to a canonical basis.