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Linear Systems

> "Linear algebra is a fantastic subject. On the one hand it is clean
> and beautiful."
>
> – Gilbert Strang, *Linear Algebra and its Applications*

This wonderful branch of mathematics is both beautiful and useful, and is
the cornerstone upon which this course is built.

Vectors

At the heart of linear algebra is machinery for solving linear equations. In
the simplest case, the number of unknowns equals the number of equations.
For example, here are two equations in two unknowns:

$$
\begin{aligned}
2x - y &= 1 \\
x + y &= 5.
\end{aligned}
\tag{1}
$$

There are at least two ways in which we can think of solving these equations
for x and y. The first is to consider each equation as describing a line, with
the solution being at the intersection of the lines: in this case the point $(2, 3)$,
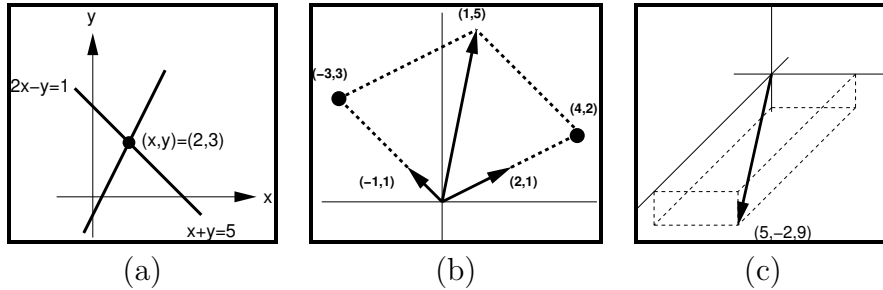Figure 1(a). This solution is termed a *row* solution because the equations

Figure 1: (a) Row solution in 2-D; (b) column solution in 2-D; and (c) column solution in 3-D.

are considered in isolation of one another. This is in contrast to a *column* solution in which the equations are rewritten in vector form:

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} x + \begin{pmatrix} -1 \\ 1 \end{pmatrix} y \;=\; \begin{pmatrix} 1 \\ 5 \end{pmatrix}. \tag{2}$$

The solution reduces to finding values for x and y that scale the vectors $(2, 1)$ and $(-1, 1)$ so that their sum is equal to the vector $(1, 5)$, Figure 1(b). Of course the solution is again $x = 2$ and $y = 3$.

These solutions generalize to higher dimensions. Here is an example with $n = 3$ unknowns and equations:

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v + 0w &= -2 \\ -2u + 7v + 2w &= 9. \end{aligned} \tag{3}$$

Each equation now corresponds to a plane, and the row solution corresponds to the intersection of the planes (i.e., the intersection of two planes is a line, and that line intersects the third plane at a point: in this case, the point $u = 1$, $v = 1$, $w = 2$). In vector form, the equations take the form:

$$\begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} u + \begin{pmatrix} 1 \\ -6 \\ 7 \end{pmatrix} v + \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} w \;=\; \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix}. \tag{4}$$

The solution again amounts to finding values for u, v, and w that scale the vectors on the left so that their sum is equal to the vector on the right, Figure 1(c).

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In the context of solving linear equations we have introduced the notion of a vector, scalar multiplication of a vector, and vector sum. In its most general form, a n-dimensional column vector is represented as:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \tag{5}$$

and a n-dimensional row vector as:

$$\vec{y} = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}. \tag{6}$$

Scalar multiplication of a vector $\vec{x}$ by a scalar value c, scales the length of the vector by an amount c, Figure 1(b), and is given by:

$$c\vec{v} = \begin{pmatrix} cv_1 \\ \vdots \\ cv_n \end{pmatrix}. \tag{7}$$

The vector sum $\vec{w} = \vec{x} + \vec{y}$ is computed via the parallelogram construction or by "stacking" the vectors head to tail, Figure 1(b), and is computed by a pairwise addition of the individual vector components:

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}. \tag{8}$$

The linear combination of vectors by vector addition and scalar multiplication is one of the central ideas in linear algebra.

Matrices

In solving n linear equations in n unknowns there are three quantities to consider. For example consider again the following set of equations:

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v + 0w &= -2 \\ -2u + 7v + 2w &= 9. \end{aligned} \tag{9}$$

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On the right is the column vector:

$$\begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix},$$ (10)

and on the left are the three unknowns that can also be written as a column vector:

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$ (11)

The set of nine coefficients (3 rows, 3 columns) can be written in matrix form:

$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}$$ (12)

Matrices, like vectors, can be added and scalar multiplied. Not surprising, since we may think of a vector as a skinny matrix: a matrix with only one column. Consider the following 3×3 matrix:

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}.$$ (13)

The matrix cA, where c is a scalar value, is given by:

$$cA = \begin{pmatrix} ca_1 & ca_2 & ca_3 \\ ca_4 & ca_5 & ca_6 \\ ca_7 & ca_8 & ca_9 \end{pmatrix}.$$ (14)

And the sum of two matrices, $A = B + C$, is given by:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} = \begin{pmatrix} b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \\ b_4 + c_4 & b_5 + c_5 & b_6 + c_6 \\ b_7 + c_7 & b_8 + c_8 & b_9 + c_9 \end{pmatrix}.$$ (15)

With the vector and matrix notation we can rewrite the three equations in the more compact form of $A\vec{x} = \vec{b}$:

$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix}.$$ (16)

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Where the multiplication of the matrix A with vector $\vec{x}$ must be such that the three original equations are reproduced. The first component of the product comes from multiplying the first row of A (a row vector) with the column vector $\vec{x}$ as follows:

$$\begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2u + 1v + 1w \end{pmatrix}. \tag{17}$$

This quantity is equal to 5, the first component of $\vec{b}$, and is simply the first of the three original equations. The full product is computed by multiplying each row of the matrix A with the vector $\vec{x}$ as follows:

$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2u + 1v + 1w \\ 4u - 6v + 0w \\ -2u + 7v + 2w \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix}. \tag{18}$$

In its most general form the product of a $m \times n$ matrix with a n dimensional column vector is a m dimensional column vector whose i^{th} component is:

$$\sum_{j=1}^{n} a_{ij} x_j, \tag{19}$$

where a_{ij} is the matrix component in the ith row and j^{th} column. The sum along the ith row of the matrix is referred to as the *inner product* or *dot product* between the matrix row (itself a vector) and the column vector $\vec{x}$. Inner products are another central idea in linear algebra. The computation for multiplying two matrices extends naturally from that of multiplying a matrix and a vector. Consider for example the following 3×4 and 4×2 matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix}. \tag{20}$$

The product $C = AB$ is a 3×2 matrix given by:

$$\begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{14}b_{41} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} + a_{14}b_{42} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} + a_{24}b_{41} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} + a_{24}b_{42} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} + a_{34}b_{41} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} + a_{34}b_{42} \end{pmatrix}. \tag{21}$$

That is, the i, j component of the product C is computed from an inner product of the ith row of matrix A and the jth column of matrix B. Notice that this definition is completely consistent with the product of a matrix and vector. In order to multiply two matrices A and B (or a matrix and a vector), the column dimension of A must equal the row dimension of B. In other words if A is of size $m \times n$, then B must be of size $n \times p$ (the product is of size $m \times p$). This constraint immediately suggests that matrix multiplication is not commutative: usually $AB \neq BA$. However matrix multiplication is both associative $(AB)C = A(BC)$ and distributive $A(B + C) = AB + AC$.

The *identity matrix* I is a special matrix with 1 on the diagonal and zero elsewhere:

$$I \;=\; \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}. \tag{22}$$

Given the definition of matrix multiplication, it is easily seen that for any vector $\vec{x}$, $I\vec{x} = \vec{x}$, and for any suitably sized matrix, $IA = A$ and $BI = B$.

In the context of solving linear equations we have introduced the notion of a vector and a matrix. The result is a compact notation for representing linear equations, $A\vec{x} = \vec{b}$. Multiplying both sides by the matrix inverse A^{-1} yields the desired solution to the linear equations:

$$\begin{aligned} A^{-1}A\vec{x} &= A^{-1}\vec{b} \\ I\vec{x} &= A^{-1}\vec{b} \\ \vec{x} &= A^{-1}\vec{b} \end{aligned} \tag{23}$$

A matrix A is invertible if there exists[1] a matrix B such that $BA = I$ and $AB = I$, where I is the identity matrix. The matrix B is the inverse of A and is denoted as A^{-1}. Note that this commutative property limits the discussion of matrix inverses to square matrices.

Not all matrices have inverses. Let's consider some simple examples. The inverse of a 1×1 matrix $A = \begin{pmatrix} a \end{pmatrix}$ is $A^{-1} = \begin{pmatrix} 1/a \end{pmatrix}$; but the inverse does not exist when $a = 0$. The inverse of a 2×2 matrix can be calculated as:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \;=\; \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \tag{24}$$

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[1]The inverse of a matrix is unique: assume that B and C are both the inverse of matrix A, then by definition $B = B(AC) = (BA)C = C$, so that B must equal C.

but does not exist when $ad - bc = 0$. Any diagonal matrix is invertible:

$$A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} 1/a_1 & & \\ & \ddots & \\ & & 1/a_n \end{pmatrix}, \qquad (25)$$

as long as all the diagonal components are non-zero. The inverse of a product of matrices AB is $(AB)^{-1} = B^{-1}A^{-1}$. This is easily proved using the associativity of matrix multiplication.[2] The inverse of an arbitrary matrix, if it exists, can itself be calculated by solving a collection of linear equations. Consider for example a 3×3 matrix A whose inverse we know must satisfy the constraint that $AA^{-1} = I$:

$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} \begin{pmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{pmatrix} = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (26)$$

This matrix equation can be considered a column at a time yielding a system of three equations $A\vec{x}_1 = \vec{e}_1$, $A\vec{x}_2 = \vec{e}_2$, and $A\vec{x}_3 = \vec{e}_3$. These can be solved independently for the columns of the inverse matrix, or simultaneously using the *Gauss-Jordan method*.

A system of linear equations $A\vec{x} = \vec{b}$ can be solved by simply left multiplying with the matrix inverse A^{-1} (if it exists). We must naturally wonder the fate of our solution if the matrix is not invertible. The answer to this question is explored later. But before moving forward we need one last definition.

The *transpose* of a matrix A, denoted as A^T, is constructed by placing the ith row of A into the ith column of A^T. For example:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 4 & -6 & 0 \end{pmatrix} \quad \text{and} \quad A^T = \begin{pmatrix} 1 & 4 \\ 2 & -6 \\ 1 & 0 \end{pmatrix}. \qquad (27)$$

In general, the transpose of a $m \times n$ matrix is a $n \times m$ matrix with $(A^T)_{ij} = A_{ji}$. The transpose of a sum of two matrices is the sum of the transposes: $(A + B)^T = A^T + B^T$. The transpose of a product of two matrices has the familiar form $(AB)^T = B^T A^T$. And the transpose of the inverse is the

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[2]In order to prove $(AB)^{-1} = B^{-1}A^{-1}$, we must show $(AB)(B^{-1}A^{-1}) = I$: $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$, and that $(B^{-1}A^{-1})(AB) = I$: $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$.

inverse of the transpose: $(A^{-1})^T = (A^T)^{-1}$. Of particular interest will be the class of *symmetric* matrices that are equal to their own transpose $A^T = A$. Symmetric matrices are necessarily square, here is a 3×3 symmetric matrix:

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 1 & -6 & 0 \\ 4 & 0 & 3 \end{pmatrix}, \tag{28}$$

notice that, by definition, $a_{ij} = a_{ji}$.

Vector Spaces

The most common *vector space* is that defined over the reals, denoted as R^n. This space consists of all column vectors with n real-valued components, with rules for vector addition and scalar multiplication. A vector space has the property that the addition and multiplication of vectors always produces vectors that lie within the vector space. In addition, a vector space must satisfy the following properties, for any vectors $\vec{x}$, $\vec{y}$, $\vec{z}$, and scalar c:

1. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
2. $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$
3. there exists a unique "zero" vector $\vec{0}$ such that $\vec{x} + \vec{0} = \vec{x}$
4. there exists a unique "inverse" vector $-\vec{x}$ such that

   $\vec{x} + (-\vec{x}) = \vec{0}$
5. $1\vec{x} = \vec{x}$
6. $(c_1 c_2)\vec{x} = c_1(c_2 \vec{x})$
7. $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$
8. $(c_1 + c_2)\vec{x} = c_1 \vec{x} + c_2 \vec{x}$

Vector spaces need not be finite dimensional, R^∞ is a vector space. Matrices can also make up a vector space. For example the space of 3×3 matrices can be thought of as R^9 (imagine stringing out the nine components of the matrix into a column vector).

A *subspace* of a vector space is a non-empty subset of vectors that is *closed* under vector addition and scalar multiplication. That is, the following constraints are satisfied: (1) the sum of any two vectors in the subspace remains in the subspace; (2) multiplication of any vector by a scalar yields a vector in the subspace. With the closure property verified, the eight properties of a vector space automatically hold for the subspace.

Example: Consider the set of all vectors in R^2 whose components are greater than or equal to zero. The sum of any two vectors in this space remains in the space, but multiplication of, for example, the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ by -1 yields the vector $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$ which is no longer in the space. Therefore, this collection of vectors does not form a vector space.

Vector subspaces play a critical role in understanding systems of linear equations of the form $A\vec{x} = \vec{b}$. Consider for example the following system:

$$\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}. \tag{29}$$

Unlike the earlier system of equations, this system is *over-constrained*, there are more equations (three) than unknowns (two). A solution to this system exists if the vector $\vec{b}$ lies in the subspace of the columns of matrix A. To see why this is so, we rewrite the above system according to the rules of matrix multiplication yielding an equivalent form:

$$x_1 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + x_2 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}. \tag{30}$$

In this form, we see that a solution exists when the scaled columns of the matrix sum to the vector $\vec{b}$. This is simply the closure property necessary for a vector subspace.

The vector subspace spanned by the columns of the matrix A is called the *column space* of A. It is said that a solution to $A\vec{x} = \vec{b}$ exists if and only if the vector $\vec{b}$ lies in the column space of A.

Example: Consider the following over-constrained system:

$$A\vec{x} = \vec{b}$$
$$\begin{pmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

The column space of A is the plane spanned by the vectors $\begin{pmatrix} 1 & 5 & 2 \end{pmatrix}^T$ and $\begin{pmatrix} 0 & 4 & 4 \end{pmatrix}^T$. Therefore, the solution $\vec{b}$ cannot be

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an arbitrary vector in $\mathcal{R}^3$, but is constrained to lie in the plane spanned by these two vectors.

At this point we have seen three seemingly different classes of linear equations of the form $A\vec{x} = \vec{b}$, where the matrix A is either:

1. square and invertible (*non-singular*),
2. square but not invertible (*singular*),
3. over-constrained.

In each case solutions to the system exist if the vector $\vec{b}$ lies in the column space of the matrix A. At one extreme is the invertible $n \times n$ square matrix whose solutions may be any vector in the whole of $\mathcal{R}^n$. At the other extreme is the zero matrix $A = 0$ with only the zero vector in it's column space, and hence the only possible solution. In between are the singular and over-constrained cases, where solutions lie in a subspace of the full vector space.

The second important vector space is the *nullspace* of a matrix. The vectors that lie in the nullspace of a matrix consist of all solutions to the system $A\vec{x} = \vec{0}$. The zero vector is always in the nullspace.

Example: Consider the following system:

$$A\vec{x} = \vec{0}$$
$$\begin{pmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The nullspace of A contains the zero vector $\begin{pmatrix} u & v & w \end{pmatrix}^T = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T$. Notice also that the third column of A is the sum of the first two columns, therefore the nullspace of A also contains all vectors of the form $\begin{pmatrix} u & v & w \end{pmatrix}^T = \begin{pmatrix} c & c & -c \end{pmatrix}^T$ (i.e., all vectors lying on a one-dimensional line in $\mathcal{R}^3$).

Basis

Recall that if the matrix A in the system $A\vec{x} = \vec{b}$ is invertible, then left multiplying with A^{-1} yields the desired solution: $\vec{x} = A^{-1}\vec{b}$. In general it is said that a $n \times n$ matrix is invertible if it has *rank n* or is *full rank*, where the

rank of a matrix is the number of *linearly independent* rows in the matrix. Formally, a set of vectors $\vec{u}_1, \vec{u}_2, ..., \vec{u}_n$ are *linearly independent* if:

$$c_1\vec{u}_1 + c_2\vec{u}_2 + ... + c_n\vec{u}_n = \vec{0} \qquad (31)$$

is true only when $c_1 = c_2 = ... = c_n = 0$. Otherwise, the vectors are *linearly dependent*. In other words, a set of vectors are linearly dependent if at least one of the vectors can be expressed as a sum of scaled copies of the remaining vectors.

Linear independence is easy to visualize in lower-dimensional subspaces. In 2-D, two vectors are linearly dependent if they lie along a line, as shown on the right. That is, there is a non-trivial combination of the vectors that yields the zero vector. In 2-D, any three vectors are guaranteed to be linearly dependent. For example, as shown on the right, the vector $\begin{pmatrix} -1 & 2 \end{pmatrix}$ can be expressed as a sum of the remaining linearly independent vectors: $\frac{3}{2}\begin{pmatrix} -2 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 2 \end{pmatrix}$. In 3-D, three vectors are linearly dependent if they lie in the same plane. Also in 3-D, any four vectors are guaranteed to be linearly dependent.

Linear independence is directly related to the nullspace of a matrix. Specifically, the columns of a matrix are linearly independent (i.e., the matrix is full rank) if the matrix nullspace contains only the zero vector. For example, consider the following system of linear equations:

$$\begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \qquad (32)$$

Recall that the nullspace contains all vectors $\vec{x}$ such that $x_1\vec{u} + x_2\vec{v} + x_3\vec{w} = 0$. Notice that this is also the condition for linear independence. If the only solution is the zero vector then the vectors are linearly independent and the matrix is full rank and invertible.

Linear independence is also related to the column space of a matrix. If the column space of a $n \times n$ matrix is all of $\mathcal{R}^n$, then the matrix is full rank.
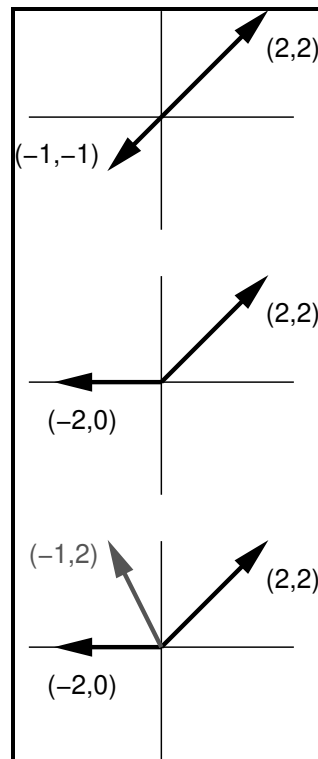
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For example, consider the following system of linear equations:

$$
\begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}. \tag{33}
$$

If the the matrix is full rank, then the solution $\vec{b}$ can be any vector in $\mathcal{R}^3$. In such cases, the vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ are said to *span* the space.

Now, a *linear basis* of a vector space is a set of linearly independent vectors that span the space. Both conditions are important. Given an n dimensional vector space with n basis vectors $\vec{v}_1, ..., \vec{v}_n$, any vector $\vec{u}$ in the space can be written as a linear combination of these n vectors:

$$
\vec{u} = a_1 \vec{v_1} + ... + a_n \vec{v_n}. \tag{34}
$$

In addition, the linear independence guarantees that this linear combination is unique. If there is another combination such that:

$$
\vec{u} = b_1 \vec{v_1} + ... + b_n \vec{v_n}, \tag{35}
$$

then the difference of these two representations yields

$$
\begin{aligned}
\vec{0} &= (a_1 - b_1)\vec{v_1} + ... + (a_n - b_n)\vec{v_n} \\
&= c_1 \vec{v_1} + ... + c_n \vec{v_n}
\end{aligned} \tag{36}
$$

which would violate the linear independence condition. While the representation is unique, the basis is not. A vector space has infinitely many different bases. For example in $\mathcal{R}^2$ any two vectors that do not lie on a line form a basis, and in $\mathcal{R}^3$ any three vectors that do not lie in a common plane or line form a basis.

> **Examples:** (1) The vectors $\begin{pmatrix} 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \end{pmatrix}$ form the canonical basis for $\mathcal{R}^2$. These vectors are both linearly independent and span the entire vector space; (2) The vectors $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 & 0 \end{pmatrix}$ do not form a basis for $\mathcal{R}^3$. These vectors lie in a 2-D plane and do not span the entire vector space; and (3) The vectors $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 2 \end{pmatrix}$, and $\begin{pmatrix} 1 & -1 & 0 \end{pmatrix}$ do not form a basis. Although these vectors span the vector space, the fourth vector is linearly dependent on the first two. Removing the fourth vector leaves a basis for $\mathcal{R}^3$.