

Master of Information and Data Science
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Professor Hany Farid
University of California, Berkeley

Linear Time Invariant Systems

A *discrete-time signal* is represented as a sequence of numbers, f , where the x th number in the sequence is denoted as $f(x)$:

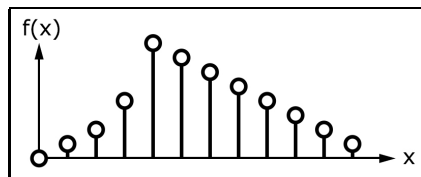
$$f = \{f(x)\}, \quad -\infty < x < \infty, \quad (1)$$

where x is an integer. Note that from this definition, a discrete-time signal is defined only for integer values of x .

For example, the finite-length sequence shown below is represented by the following sequence of numbers

$$\begin{aligned} f &= \{f(1) \ f(2) \ \dots \ f(n)\} \\ &= \{0 \ 1 \ 2 \ 4 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2 \ 1\}. \end{aligned} \quad (2)$$

For notational convenience, we will drop the cumbersome notation of Equation (1), and refer to the entire sequence simply as $f(x)$. Discrete-time signals often arise from the periodic sampling of continuous-time (analog) signals, a process that we will discuss later.

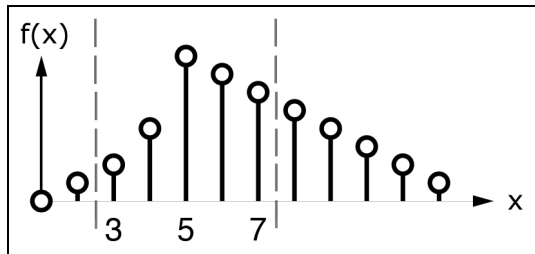


Discrete-Time Systems: In its most general form, a *discrete-time system* is a transformation that maps a discrete-time signal, $f(x)$, onto a unique $g(x)$, and is denoted as:

$$g(x) = T\{f(x)\} \quad (3)$$

Example: Consider the following system:

$$g(i) = \frac{1}{2N + 1} \sum_{k=-N}^N f(i + k).$$



In this system, the i th number in the output sequence is computed as the average of $2N + 1$ elements centered around the i th input element. As shown above, with $N = 2$, the output value at $x = 5$ is computed as $1/5$ times the sum of the five input elements between the dotted lines. Subsequent output values are computed by sliding these lines to the right.

Although in the above example, the output at each position depended on only a small number of input values, in general, this may not be the case, and the output may be a function of all input values.

Linear Time-Invariant Systems: Of particular interest to us will be a class of discrete-time systems that are both *linear* and *time-invariant*. A system is said to be linear if it obeys the rules of superposition, namely:

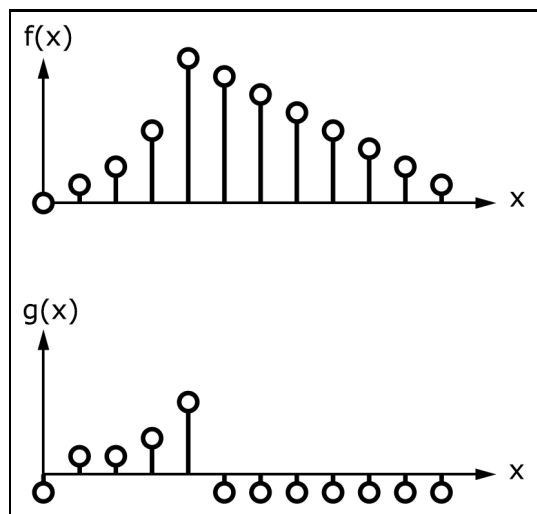
$$T\{af_1(x) + bf_2(x)\} = aT\{f_1(x)\} + bT\{f_2(x)\}, \quad (4)$$

for any constants a and b . A system, $T\{\cdot\}$ that maps $f(x)$ onto $g(x)$ is shift- or time-invariant if a shift in the input causes a similar shift in the output:

$$g(x) = T\{f(x)\} \implies g(x - x_0) = T\{f(x - x_0)\}. \quad (5)$$

Example: Consider the following system:

$$g(x) = f(x) - f(x - 1), \quad -\infty < x < \infty.$$



In this system, the k th number in the output sequence is computed as the difference between the k th and k th-1 elements in the input sequence. Is this system linear? We need only show that this system obeys the principle of superposition:

$$\begin{aligned}
 T\{af_1(x) + bf_2(x)\} &= (af_1(x) + bf_2(x)) - (af_1(x-1) + bf_2(x-1)) \\
 &= (af_1(x) - af_1(x-1)) + (bf_2(x) - bf_2(x-1)) \\
 &= a(f_1(x) - f_1(x-1)) + b(f_2(x) - f_2(x-1)) \\
 &= aT\{f_1(x)\} + bT\{f_2(x)\}
 \end{aligned}$$

which, according to Equation (4), makes $T\{\cdot\}$ linear. Is this system time-invariant? First, consider the shifted signal, $f_1(x) = f(x - x_0)$, then:

$$g_1(x) = f_1(x) - f_1(x-1) = f(x - x_0) - f(x - 1 - x_0),$$

and,

$$g(x - x_0) = f(x - x_0) - f(x - 1 - x_0) = g_1(x),$$

so that this system is time-invariant.

Example: Consider the following system:

$$g(x) = f(nx), \quad -\infty < x < \infty,$$

where n is a positive integer. This system creates an output sequence by selecting every n th element of the input sequence. Is this system linear?

$$T\{af_1(x) + bf_2(x)\} = af_1(nx) + bf_2(nx)$$

which, according to Equation (4), makes $T\{\cdot\}$ linear. Is this system? First, consider the shifted signal, $f_1(x) = f(x - x_0)$, then:

$$g_1(x) = f_1(nx) = f(nx - x_0),$$

but,

$$g(x - x_0) = f(n(x - x_0)) \neq g_1(x),$$

so that this system is not time-invariant.

The precise reason why we are particularly interested in linear time-invariant systems will become more clear in a few sections. But before pressing on, the concept of discrete-time systems is reformulated within a linear-algebraic framework. In order to accomplish this, it is necessary to first restrict ourselves to consider input signals of finite length. Then, any discrete-time linear system can be represented as a matrix operation of the form:

$$\vec{g} = M\vec{f}, \tag{6}$$

where, \vec{f} is the input signal, \vec{g} is the output signal, and the matrix M embodies the discrete-time linear system.

Example: Consider the following system:

$$g(x) = f(x - 1), \quad 1 < x < 8.$$

The output of this system is a shifted copy of the input signal, and can be formulated in matrix notation as:

$$\begin{pmatrix} g(1) \\ g(2) \\ g(3) \\ g(4) \\ g(5) \\ g(6) \\ g(7) \\ g(8) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} f(1) \\ f(2) \\ f(3) \\ f(4) \\ f(5) \\ f(6) \\ f(7) \\ f(8) \end{pmatrix}$$

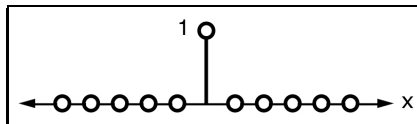
Note that according to the initial definition of the system, the output signal at $x = 1$ is undefined (i.e., $g(1) = f(0)$). In the above matrix formulation we have adopted a common solution to this problem by considering the signal as wrapping around itself and setting $g(1) = f(8)$.

Any system expressed in the matrix notation of Equation (6) is a discrete-time linear system, but not necessarily a time-invariant system. But, if we constrain ourselves to *Toeplitz* matrices, then the resulting expression will be a linear time-invariant system. A Toeplitz matrix is one in which each row contains a shifted copy of the previous row. For example, a 5×5 Toeplitz matrix is of the form

$$M = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 & m_5 \\ m_5 & m_1 & m_2 & m_3 & m_4 \\ m_4 & m_5 & m_1 & m_2 & m_3 \\ m_3 & m_4 & m_5 & m_1 & m_2 \\ m_2 & m_3 & m_4 & m_5 & m_1 \end{pmatrix} \quad (7)$$

It is important to feel comfortable with this formulation because later concepts will be built upon this linear algebraic framework.

Convolution: In the previous section, a discrete-time signal was represented as a sequence of numbers. More formally, this representation is in terms of the discrete-time *unit impulse* shown to the right and defined as:



$$\delta(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0. \end{cases} \quad (8)$$

Any discrete-time signal can be represented as a sum of scaled and shifted unit-impulses:

$$f(x) = \sum_{k=-\infty}^{\infty} f(k)\delta(x - k), \quad (9)$$

where the shifted impulse $\delta(x - k) = 1$ when $x = k$, and is zero elsewhere.

Example: Consider the following discrete-time signal, centered at $x = 0$.

$$f(x) = (\dots 0 \ 0 \ 2 \ -1 \ 4 \ 0 \ 0 \ \dots),$$

this signal can be expressed as a sum of scaled and shifted unit-impulses:

$$\begin{aligned} f(x) &= 2\delta(+1) - 1\delta(0) + 4\delta(-1) \\ &= f(-1)\delta(+1) + f(0)\delta(0) + f(1)\delta(-1) \\ &= \sum_{k=-1}^1 f(k)\delta(-k). \end{aligned}$$

Let's now consider what happens when we present a linear time-invariant system with this new representation of a discrete-time signal:

$$\begin{aligned} g(x) &= T\{f(x)\} \\ &= T\left\{\sum_{k=-\infty}^{\infty} f(k)\delta(x-k)\right\}. \end{aligned} \tag{10}$$

By the property of linearity, Equation (4), the above expression may be rewritten as:

$$g(x) = \sum_{k=-\infty}^{\infty} f(k)T\{\delta(x-k)\}. \tag{11}$$

Imposing the property of time-invariance, Equation (5), if $h(x)$ is the response to the unit-impulse, $\delta(x)$, then the response to $\delta(x-k)$ is simply $h(x-k)$. And now, the above expression can be rewritten as:

$$g(x) = \sum_{k=-\infty}^{\infty} f(k)h(x-k). \tag{12}$$

Consider for a moment the implications of the above equation. The unit-impulse response, $h(x) = T\{\delta(x)\}$, of a linear time-invariant system fully characterizes that system. More precisely, given the unit-impulse response, $h(x)$, the output, $g(x)$, can be determined for *any* input, $f(x)$.

The sum in Equation (12) is commonly called the *convolution sum* and may be expressed more compactly as:

$$g(x) = f(x) \star h(x). \quad (13)$$

A more mathematically correct notation is $(f \star h)(x)$, but for later notational considerations, we will adopt the above notation.

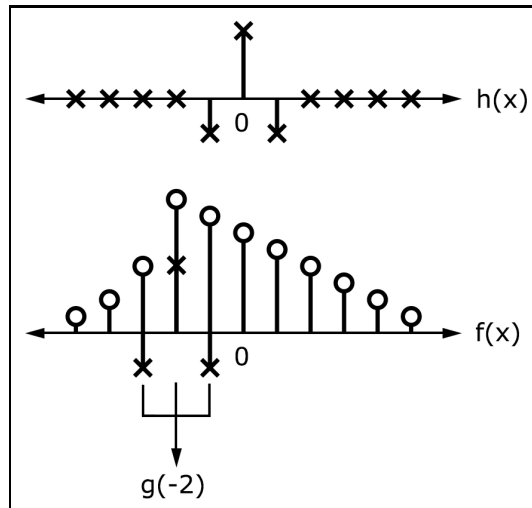
Example: Consider the following finite-length unit-impulse response:

$$h(x) = (-2 \ 4 \ -2),$$

and the input signal, $f(x)$, shown above. Then the output signal at, for example, $x = -2$, is computed as:

$$\begin{aligned} g(-2) &= \sum_{k=-3}^{-1} f(k)h(-2-k) \\ &= f(-3)h(1) + f(-2)h(0) + f(-1)h(-1). \end{aligned}$$

The next output sum at $x = -1$, is computed by “sliding” the unit-impulse response along the input signal and computing a similar sum.



Since linear time-invariant systems are fully characterized by convolution with the unit-impulse response, properties of such systems can be determined by considering properties of the convolution operator. For example, convolution is *commutative*:

$$\begin{aligned}
 f(x) \star h(x) &= \sum_{k=-\infty}^{\infty} f(k)f(x-k), \quad \text{let } j = x - k \\
 &= \sum_{j=-\infty}^{\infty} f(x-j)h(j) = \sum_{j=-\infty}^{\infty} h(j)f(x-j) \\
 &= h(x) \star f(x).
 \end{aligned} \tag{14}$$

Convolution also *distributes* over addition:

$$\begin{aligned}
 f(x) \star (h_1(x) + h_2(x)) &= \sum_{k=-\infty}^{\infty} f(k)(h_1(x-k) + h_2(x-k)) \\
 &= \sum_{k=-\infty}^{\infty} f(k)h_1(x-k) + f(k)h_2(x-k) \\
 &= \sum_{k=-\infty}^{\infty} f(k)h_1(x-k) + \sum_{k=-\infty}^{\infty} f(k)h_2(x-k) \\
 &= f(x) \star h_1(x) + f(x) \star h_2(x).
 \end{aligned} \tag{15}$$

A final useful property of linear time-invariant systems is that a cascade of systems can be combined into a single system with impulse response equal to the convolution of the individual impulse responses. For example, for a cascade of two systems:

$$(f(x) \star h_1(x)) \star h_2(x) = f(x) \star (h_1(x) \star h_2(x)). \tag{16}$$

This property is fairly straight-forward to prove, and offers a good exercise in manipulating the convolution sum:

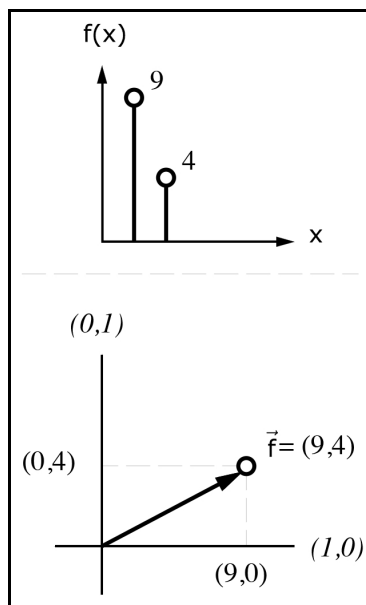
$$\begin{aligned}
 g_1(x) &= f(x) \star h_1(x) \\
 &= \sum_{k=-\infty}^{\infty} f(k)h_1(x-k)
 \end{aligned} \tag{17}$$

and

$$\begin{aligned}
g_2(x) &= (f(x) \star h_1(x)) \star h_2(x) \\
&= g_1(x) \star h_2(x) \\
&= \sum_{j=-\infty}^{\infty} g_1(j)h_2(x-j) \quad \text{substituting for } g_1(x), \\
&= \sum_{j=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} f(k)h_1(j-k) \right) h_2(x-j) \\
&= \sum_{k=-\infty}^{\infty} f(k) \left(\sum_{j=-\infty}^{\infty} h_1(j-k)h_2(x-j) \right) \quad \text{let } i = j - k, \\
&= \sum_{k=-\infty}^{\infty} f(k) \left(\sum_{i=-\infty}^{\infty} h_1(i)h_2(x-i-k) \right) \\
&= f(x) \star (h_1(x) \star h_2(x)). \tag{18}
\end{aligned}$$

Let's consider now how these concepts fit into the linear-algebraic framework. First, a length N signal can be thought of as a point in a N -dimensional vector space. As a simple example, consider the length 2 signal shown on the right, represented as a vector in a 2-dimensional space.

Earlier, the signal $f(x)$ was expressed as a sum of weighted and shifted impulses, $f(x) = 9\delta(x) + 4\delta(x-1)$, and in the vector space, it is expressed with respect to the canonical basis as $\vec{f} = 9(1 \ 0) + 4(0 \ 1)$. The weighting of each basis vector is determined by simply projecting the vector \vec{f} onto each axis. With this vector representation, it is then natural to express the convolution sum (i.e., a linear time-invariant system) as a matrix operation. For example, let $h(x) = (h_{-1} \ h_0 \ h_1)$ be the finite-length unit-impulse response of a linear time-invariant system, $T\{\cdot\}$, then the system $g(x) = T\{f(x)\}$ can



be expressed as $\vec{g} = M\vec{f}$, where the matrix M is of the form:

$$M = \begin{pmatrix} h_0 & h_{-1} & 0 & 0 & \dots & 0 & 0 & 0 & h_1 \\ h_1 & h_0 & h_{-1} & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & h_1 & h_0 & h_{-1} & \dots & 0 & 0 & 0 & 0 \\ \vdots & & & & \ddots & & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & h_1 & h_0 & h_{-1} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & h_1 & h_0 & h_{-1} \\ h_{-1} & 0 & 0 & 0 & \dots & 0 & 0 & h_1 & h_0 \end{pmatrix}, \quad (19)$$

where each row contains a shifted and time-reversed copy of the unit-impulse response, $h(x)$. The convolution matrix can then be thought of as simply transforming the basis set. As expected, this matrix is a Toeplitz matrix of the form given earlier in Equation (7). The reason for the time-reversal can be seen directly from the convolution sum of Equation (12). More specifically, the output signal $g(x)$ at a fixed x , is determined by summing the products of $f(k)h(x-k)$ for all k . Note that the signal $h(x-k)$ can be equivalently written as $h(-k+x)$, which is a shifted (by x) and time-reversed (because of the $-k$) copy of the impulse response. Note also that when expressed in matrix form, it becomes immediately clear that the convolution sum is invertible, when h is not identically zero: $\vec{g} = M\vec{f}$ and $\vec{f} = M^{-1}\vec{g}$.

Before pressing on, let's try to combine the main ideas seen so far into a single example. We will begin by defining a simple discrete-time system, show that it is both linear and time-invariant, and compute its unit-impulse response

Example: Define the discrete-time system, $T\{\cdot\}$ as:

$$g(x) = f(x+1) - f(x-1).$$

This system is linear because it obeys the rule of superposition:

$$\begin{aligned} T\{af_1(x) + bf_2(x)\} &= (af_1(x+1) + bf_2(x+1)) - (af_1(x-1) + bf_2(x-1)) \\ &= (af_1(x+1) - af_1(x-1)) + (bf_2(x+1) - bf_2(x-1)) \\ &= a(f_1(x+1) - f_1(x-1)) + b(f_2(x+1) - f_2(x-1)) \end{aligned}$$

This system is also time-invariant because a shift in the input, $f_1(x) = f(x-x_0)$, leads to a shift in the output:

$$\begin{aligned} g_1(x) &= f_1(x+1) - f_1(x-1) \\ &= f(x+1-x_0) - f(x-1-x_0) \quad \text{and,} \\ g(x-x_0) &= f(x+1-x_0) - f(x-1-x_0) \\ &= g_1(x). \end{aligned}$$

The unit-impulse response is given by:

$$\begin{aligned} h(x) &= T\{\delta(x)\} \\ &= \delta(x+1) - \delta(x-1) \\ &= (\dots \ 0 \ 1 \ 0 \ -1 \ 0 \ \dots). \end{aligned}$$

So, convolving the finite-length impulse response $h(x) = (1 \ 0 \ -1)$ with any input signal, $f(x)$, gives the output of the linear time-invariant system, $g(x) = T\{f(x)\}$:

$$g(x) = \sum_{k=-\infty}^{\infty} f(k)h(x-k) = \sum_{k=x-1}^{x+1} f(k)h(x-k).$$

And, in matrix form, this linear time-invariant system is given by $\vec{g} = M\vec{f}$, where:

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & & & & \ddots & & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 0 \end{pmatrix}.$$